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# NONSYMMETRIC BENDING OF A PLATE REINFORCED BY A SYMMETRIC SYSTEM OF RADIAL RIBS*

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A solution is given in quadratures for the bending problem of an elastic plate reinforced cyclically by a symmetric system of arbitrarily loaded radial elastic rib-stiffeners; the bending stiffness of the ribs varies as a power of the radius and can be constant in particular. The limit case is considered for this problem, the bending of a plate a star system of finite absolutely rigid ribs.

The reinforcing elements of thin-walled structures (belts, straps, stiffeners) are ordinarily arranged in a regular manner. However, nonregular loads induce asymmetry in the state of stress and strain of a strucutre and methods of solving periodic problems become inapplicable. Most interesting results on taking account of arbitrary symmetry of the domains in boundary value problems are obtained in /1,2/. As is noted in /3/, boundary value problems for domains possessing cyclic, translational, screw, and spiral symmetric are solved effectively by traditional analytic methods after the application of a discrete or finite discrete Fourier transformation. Problems solvable by the method of integral transforms /3/, by the Wiener—Hopf method /4,5/, by Gakhov and Muskhelishvili methods /6/, were studied earlier by this method. This paper borders on these investigations and demonstrates the example of reducing boundary value problems for symmetric domains of the type mentioned to a Barnes difference equation.

1. Let an elastic plate $0 \leqslant r < \infty, 0 \leqslant \theta \leqslant 2\pi$ (Fig.1) of thickness $h$, Young's modulus $E$, and Poisson ratio $\nu$ be reinforced by a cyclically symmetric system of $N$ identical radial ribs, elastic rods $0 \leqslant r < \infty, \theta = 2\alpha k$ with the bending stiffness $S(r) = Sr^{\omega+1}$, where $r, \theta$ are polar coordinates, the angle $\theta$ is measured clockwise, $\alpha = \pi N^{-1}, k = 0, 1, \ldots, N-1, N \geqslant 1, S > 0, \omega \leqslant -1$. A transverse load $q_k(r)$, relative to the plate, is applied to the $k$-th rod, a transverse force $P_0$ and bending moments $M_{01}, M_{02}$ directed along the rays $\theta = {}^3/_2\pi$ and $\theta = 0$ act on the point $r = 0$. There are no moment interactions between the rods and the plate, i.e., the bending moments in the plate are continuous when going through the contact lines, there are not torsional strains of the rods or they are considered small and not taken into account, and $M_{02} = 0$ for $N \leqslant 2$. Find the deflections of the ribbed plate under the condition of its simple support at infinity.

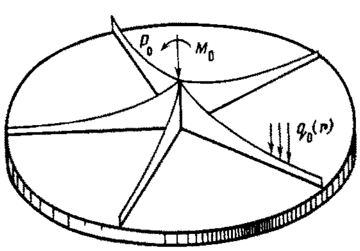


Fig.1

Let us partition the plate into $N$ identical sectors by the rays $\theta = (2k+1)\alpha,\ k = 0, 1, \ldots, N-1$ and let us introduce a polar coordinate system $r, \gamma$ with axis $\gamma = 0$ directed along the rib, into each. The components of the state of stress and strain will be denoted by the superscripts $j = 1$ for $-\alpha \leqslant \gamma \leqslant 0$ and $j = 2$ for $0 \leqslant \gamma \leqslant \alpha$. Then the deflections $w_k^j$ in the $k$-th sector should satisfy the plate bending equation

$$\Delta\Delta w_k^j(r, \gamma) = 0 \tag{1.1}$$

the conditions of contact with the stiffener ribs, the conditions of matching of the solutions in adjacent sectors, and the condition at infinity

$$w_k^{1n}(r, 0) = w_k^{2n}(r, 0),\ n = 0, 1, 2;\ w_k^{jn} \equiv \partial^n w_k^j / \partial\gamma^n \tag{1.2}$$

$$\frac{\partial^2}{\partial r^2}\left[S(r)\frac{\partial^2 w_k^1(r, 0)}{\partial r^2}\right] = N_k^2(r, 0) - N_k^1(r, 0) + q_k(r) \tag{1.3}$$

$$N_k^j = -D\left[\frac{1}{r}\frac{\partial}{\partial\gamma}(\Delta w_k^j) + (1-\nu)\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial w_k^j}{\partial\gamma}\right)\right]$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

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$$w_{k+1}^{1n}(r, -\alpha) = w_k^{2n}(r, \alpha), \quad n = 0, 1, 2, 3, \quad k = 0, 1, \ldots, N-1; \tag{1.4}$$
$$w_N^{jn} \equiv w_0^{jn}$$

$$w_k^j(r, \gamma) = \frac{P_0 + P}{8\pi D} r^2 \ln r - [(M_{01} + M_1)\cos\theta + (M_{02} + M_2)\sin\theta] \times \tag{1.5}$$
$$\frac{r \ln r}{4\pi D} + O(r); \quad P = \sum_{n=0}^{N-1} \int_0^\infty q_n(r)\,dr,$$
$$M_1 + iM_2 = \sum_{n=0}^{N-1} e^{2i\alpha n} \int_0^\infty q_n(r)\,r\,dr, \quad \omega < -1$$

Here $P$ is the principal vector, $M_1$ and $M_2$ are the principal moments of the loads $q_k(r)$ directed along the axes $\theta = {}^3/_2\pi$ and $\theta = 0$.

Condition (1.5) corresponds to the mechanically obvious fact that for all $\omega < -1$ the rib stiffness at infinity equals zero, and therefore, the states of stress of the ribbed and rib-free plates under consideration that are laoded by the force $P_0 + P$ and the moments $M_{0s} + M_s$, $s = 1, 2$ at the center as $r \to \infty$ become indistinguishable; simple support is characterized by the absence of the component $O(r^2)$ in (1.5). For $\omega = -1$ the plate remains ribbed even at infinity, and in this case condition (1.5) is not used.

**2.** Following /3/, we apply a finite discrete Fourier transformation to the problem (1.1) – (1.5)

$$g_*(r, \gamma, l) = \sum_{k=0}^{N-1} g_k(r, \gamma)\, e^{2i\alpha kl}, \quad g_k(r, \gamma) = \frac{1}{N} \sum_{l=0}^{N-1} g_*(r, \gamma, l)\, e^{-2i\alpha kl} \tag{2.1}$$

Then the problem (1.1)–(1.5) goes over into a boundary value problem for the transform $w_*^j(r, \gamma, l)$ in the cell $0 \leqslant r < \infty$, $-\alpha \leqslant \gamma \leqslant \alpha$

$$\Delta\Delta w_*^j(r, \gamma, l) = 0, \quad w_*^{1n}(r, 0, l) = w_*^{2n}(r, 0, l), \ n = 0, 1, 2 \tag{2.2}$$
$$\frac{\partial^2}{\partial r^2}\left[S(r)\frac{\partial^2 w_*^1(r, 0, l)}{\partial r^2}\right] = N_*^2(r, 0, l) - N_*^1(r, 0, l) + q_*(r, l) \tag{2.3}$$
$$w_*^{1n}(r, -\alpha, l) = e^{2i\alpha l} w_*^{2n}(r, \alpha, l), \quad n = 0, 1, 2, 3 \tag{2.4}$$
$$w_*^j(r, \gamma, l) = {}^1/_8\alpha^{-1}D^{-1}\{(P_0 + P)\,\delta_{0l} r^2 \ln r - \tag{2.5}$$
$$[(M_0 + M)\,\delta_{1l} e^{-i\gamma} + (\overline{M}_0 + \overline{M})\,\delta_{N-1,l} e^{i\gamma}]\, r \ln r + O(r)$$
$$M_0 = M_{01} + iM_{02}, \quad M = M_1 + iM_2, \quad \omega < -1$$

where $\delta_{kl}$ is the Kronecker delta.

We seek the solution of (2.2)–(2.5) in Mellin integrals that satisfy conditions (2.2) and (2.4)

$$w_*^j(r, \gamma, l) = \frac{1}{2\pi i} \int_L \frac{B(p, l)}{\sin 2(p-1)\alpha} \{\sin 2(p-1)\,\alpha \cos(p-1)\gamma - \tag{2.6}$$
$$(-1)^j \beta_j^- \sin(p-1)\gamma - \frac{(p-1)\beta^-}{(p+1)\beta^+}[\sin 2(p+1)\alpha \cos(p+1)\gamma -$$
$$(-1)^j \beta_j^+ \sin(p+1)\gamma]\} r^{1-p}\,dp + \chi(\omega) r^2$$
$$\beta_j^\pm \equiv \beta_j^\pm(p, l) = \beta^\pm(p, l) + (-1)^j \sin 2l\alpha$$
$$\beta^\pm \equiv \beta^\pm(p, l) = \cos 2(p \pm 1)\alpha - \cos 2l\alpha$$

Here $\chi(\omega) = 0$ for $\omega < -1$, $\chi(\omega) = C = \text{const}$ for $\omega = -1$, $L$ is the line $\operatorname{Re} p = \lambda$, $\lambda = a - \varepsilon$, $\varepsilon > 0$ is a sufficiently small constant, and $a$ is a numer dependent on $l$. Substituting (2.6) into (2.3), introducing the unknown function $A(p)$ by the relationships

$$A(p) \equiv A(p, l) \equiv \overline{N}_*^1(p, 0, l) - \overline{N}_*^2(p, 0, l) - \bar{q}_*(p, l) = \tag{2.7}$$
$$8Dp(1-p)\sin^{-1} 2(p-1)\alpha\beta^- B(p, l) - \bar{q}_*(p, l)$$
$$\bar{g}(p) = \int_0^\infty g(r)\, r^{p+1}\,dr$$

and following /7/, we obtain the Barnes equation

$$A(p-\omega) = F(p)A(p) + f(p), \quad p \in L \tag{2.8}$$
$$F(p) \equiv F(p,l) = \frac{Q(p-\omega)(p+1-\omega)\varphi_1(p)}{(p+1)\varphi_2(p)}, \quad Q = \frac{S}{4D}$$
$$\varphi_1(p) \equiv \varphi_1(p, l) = (p+1)\beta^+(p)\sin 2(p-1)\alpha - (p-1)\beta^-(p)\sin 2(p+1)\alpha$$
$$\varphi_2(p) \equiv \varphi_2(p, l) = 2\beta^+(p)\beta^-(p), \quad f(p) \equiv f(p, l) = F(p)\bar{q}_*(p)$$
$$\bar{q}_*(p) \equiv \bar{q}_*(p, l), \quad \beta^\pm(p) \equiv \beta^\pm(p, l)$$

for the function $A(p)$ that is analytic in the strip $\Omega = \{p \mid \lambda \leqslant \operatorname{Re} p \leqslant \lambda - \omega\}$ and tends to zero for $|\operatorname{Im} p| \to \infty$ within this strip.

3. Let us consider (2.8). We first investigate the solution of the homogeneous equation

$$X(p-\omega) = F(p)X(p), \quad p \in L \tag{3.1}$$

that is analytic in $\Omega$.

We introduce the function $Y(p)$ in the form /8/

$$Y(p) = X(p)F(p), \quad p \in \Omega^+; \; Y(p) = X(p), \quad p \in \Omega^- \tag{3.2}$$
$$\Omega^+ = \{p \mid \lambda \leqslant \operatorname{Re} p < a\}, \; \Omega^- = \{p \mid a < \operatorname{Re} p \leqslant \lambda - \omega\}$$

We denote the functions analytic in $\Omega^+$ and $\Omega^-$ by the superscripts plus and minus. If the function $F^{-1}(p)$ is analytic in the strip $\Omega^+$, then by virtue of (3.2) the function $X(p)$ will be analytic in $\Omega$ upon compliance with the equality $X(p) \equiv Y^-(p)$, $\operatorname{Re} p = a$ and the boundary condition

$$Y^+(p) = F(p)Y^-(p), \quad \operatorname{Re} p = a \tag{3.3}$$

Since it follows from (3.1) and (3.2) that $Y(p-\omega) = Y(p)$, $p \in L$, the Riemann problem (3.3) should be solved in the class of automorphic functions with the automorphicity strip $\Omega$.

Let the function $f(p)$ by analytic in $\Omega^+$. Then exactly as (3.1), the equation (2.8) will reduce to the problem of a jump

$$W^+(p) - W^-(p) = f_1(p), \quad \operatorname{Re} p = a \tag{3.4}$$

in the class of automorphic functions $W(p)$ determined by the relationships

$$Z(p) = A(p)X^{-1}(p), \quad f_1(p) = f(p)X^{-1}(p-\omega) \tag{3.5}$$
$$W(p) = Z(p) + f_1(p), \; p \in \Omega^+, \; W(p) = Z(p), \quad p \in \Omega^-$$
$$Z(p) \equiv W^-(p), \quad \operatorname{Re} p = a$$

The boundary value problems (3.3) and (3.4) are solved by quadratures by F.D. Gakhov /9/. In particular, if the index of the function $F(p)$ is zero $(\operatorname{Ind} F(p) = 0)$ on the line $\operatorname{Re} p = a$ and $\ln F(p), f_1(p) \in H$, where $H$ is the class of function satisfying the Holder condition, then according to /9/ the canonical solution of the problem (3.3) and the solution of the problem (3.4) have the form

$$Y(p) = \exp\left\{\frac{1}{2\omega i}\int_{a-i\infty}^{a+i\infty} \ln F(t)\operatorname{ctg}\frac{\pi(p-t)}{\omega}\,dt\right\} \tag{3.6}$$
$$W(p) = \frac{1}{2\omega i}\int_{a-i\infty}^{a+i\infty} f_1(t)\left[\operatorname{ctg}\frac{\pi(t-p)}{\omega} + i\right]dt$$

Following /10/, we give a more effective solution of the problem (2.8) than (3.1)–(3.6). Let $X(p)$ be an analytic solution in $\Omega$ for the problem $X(p-\omega) = -F(p)X(p)$, $p \in L$. Let us introduce the function

$$Z(p) = A(p)X^{-1}(p), \; f_1(p) = f(p)X^{-1}(p-\omega) \tag{3.7}$$
$$W(p) = Z(p) - f_1(p), \quad p \in \Omega^+; \quad W(p) = Z(p), \; p \in \Omega^-$$

Then if $f_1(p)$ is a function analytic in $\Omega^+$, then the problem (2.8) again reduces to the problem of a jump

$$W^+(p) - W^-(p) = -f_1(p), \quad \operatorname{Re} p = a \tag{3.8}$$

but with another periodicity condition: $W(p-\omega) = -W(p)$, $\operatorname{Re} p = \lambda$. If $f_1(p) \in H$ for $\operatorname{Re} p = a$, the solution of this problem has the form

$$W(p) = -\frac{1}{2\omega i} \int_{a-i\infty}^{a+i\infty} f_1(t) \sin^{-1} \frac{\pi(t-p)}{\omega}\, dt \tag{3.9}$$

and differs from (3.6) by the exponential convergence of the integral.

The equation (2.8) is reduced to a Carleman problem in /11,12/. As in /10/, another method is used below. After factorization $F(p) = F_1(p) F_2(p)$, where $F_1(p)$ is an elementary function, $\ln F_2(p) \in H$, $\operatorname{Ind} F_2(p) = 0$, the problem (3.1) with coefficient $F_1(p)$ is solved by the Barnes method, the problem (3.1) with the coefficient $F_2(p)$ and the problem (2.8) are solved by (3.2), (3.6) and (3.7), (3.9).

4. Let us investigate three case: $l = 0$, $l = 1$, $l = 2, 3, \ldots, E[{}^1/_2(N+1)]$, where $E[x]$ is the integer part of $x$, and $N \geqslant 2$. This is sufficient for finding the function $A(p, l)$ for all $l$.

Let $l = 0$. Then for a selected kernel of the Mellin transform, the highest term in the asymptotic formula (2.5) is determined by the residue of the integrand (2.6) at the point $p = -1$, therefore, $a = -1$. We put

$$F_2(p, l) = \varphi_1(p, l)\, \varphi_2^{-1}(p, l) \operatorname{tg} [{}^1/_2 \pi \omega^{-1} (p-a)] \tag{4.1}$$

According to (2.8), the asymptotic of the function $F_2(p, l)$ has the following form for large $|y|$

$$F_2(a+iy, l) = 1 + O[y \exp(-2\alpha |y|)] + O[\exp(\pi |y| \omega^{-1})] \tag{4.2}$$

The functions $\varphi_s(p, 0), s = 1, 2$ are real on the imaginary axis, $p = 0$ is a simple zero of the function $\varphi_1(p, 0)$, $p = -1$ is a simple and double zero of the functions $\varphi_1(p, 0)$ and $\varphi_2(p, 0)$; it can be shown that there are no other zeroes of the function $\varphi_s(p, 0)$ in the strip $-1 \leqslant \operatorname{Re} p \leqslant 0$. Hence, it is seen that the indices of the function $\varphi_1(p, 0)\, \varphi_2^{-1}(p, 0)$ equal $-{}^1/_2$ and ${}^1/_2$, on the lines $\operatorname{Re} p = -\varepsilon$ and $\operatorname{Re} p = -1-\varepsilon$, $\operatorname{Ind} F_2(p, 0) = 0$ on the line $\operatorname{Re} p = -1$, and $\ln F_2(p, 0) \in H$ by virtue of (4.1). In conformity with Sect.3, taking into account the conditions of the regularity and decrease of $A(p, l)$ in $\Omega$, the general solution of (2.8) can be written as

$$A(p,0) = A_0(p,0)\left[C_1 + \frac{\omega}{\pi} C_2 \operatorname{tg} \frac{\pi(p-a)}{\omega} + \frac{\omega}{\pi} \sin \frac{\pi(p-a)}{\omega} Z(p,0)\right] \tag{4.3}$$

$$A_0(p,0) = \frac{\pi}{\omega}(p+1)(-\omega Q)^{-p/\omega}\, \Gamma\left(\frac{\omega-p}{\omega}\right) \times$$
$$\operatorname{ctg} \frac{\pi(p-a)}{\omega} \cos^{-1} \frac{\pi(p-a)}{2\omega} X(p, l), \quad \omega < -1$$

$$A(p, 0) = A_0(p, 0)\,[C_1 - \cos \pi p\, Z(p, l)] \tag{4.4}$$
$$A_0(p, 0) = Q^p \Gamma(p+2) \cos^{-1} [{}^1/_2 \pi (p-a)] X(p, l), \; \omega = -1$$

$$X(p, l) = F_2^{-1}(p, l)\, Y(p, l), \quad Z(p, l) = W(p, l) - f_1(p, l) \tag{4.5}$$
$$\lambda \leqslant \operatorname{Re} p < a$$
$$X(p, l) = Y(p, l), \quad Z(p, l) = W(p, l), \; a < \operatorname{Re} p \leqslant \lambda - \omega$$

$$Y(p, l) \equiv Y(p) = \exp\left[\frac{1}{2\omega i} \int_{a-i\infty}^{a+i\infty} \ln F_2(t, l) \operatorname{ctg} \frac{\pi(t-p)}{\omega}\, dt\right]$$

$$W(p, l) \equiv W(p) = \frac{1}{2\omega i} \int_{a-i\infty}^{a+i\infty} f_1(t, l) \sin^{-1} \frac{\pi(t-p)}{\omega}\, dt$$

$$f_1(p, l) \equiv f_1(p) = A_0^{-1}(p, l) \cos^{-1} \frac{\pi(p-a)}{\omega}\, \bar{q}_*(p, l)$$

Here $a = -1, l = 0, \Gamma(p)$ is the Gamma function. The constants $C_1$ and $C_2$ are found for $\omega < -1$ from the condition (2.5) by using the asymptotic $O(r^2 \ln r) + O(r^2) + O(r)$ of the integral (2.6), which can be obtained by shifting the contour $L$ to the right after the point $p = -1$, and adding the residue at this point to (2.6). By using (2.7) we insert the function $A(p, l)$ into (2.6). The expression (4.3) shows that the factors for $C_2$ and $Z(p, l)$ have simple poles at the point $p = -1$, the factor for $C_1$ has a second-order pole. Residues of the integrand of (2.6) at the simple pole are proportional to $r^2$, the residue at the second-order pole is the sum of functions proportional to $r^2$ and $r^2 \ln r$. Equating the factors for $r^2 \ln r$ in (2.5) and in the asymptotic of the solution (2.6), (2.7), (4.3)–(4.5), we obtain

$$C_1 = P_0 A_0^{-1}(-1, 0) = -P_0 \pi^{1/2} (2\alpha)^{-1/2} Q^{-1/\omega} \times (-\omega)^{1/2 - 1/\omega} \Gamma^{-1}(\omega^{-1})\, Y^{-1}(-1) \tag{4.6}$$

Taking into account that there is no component $O(r^2)$ in (2.5), we obtain

$$C_2 = C_1\left\{\frac{\ln(-\omega Q)}{\omega} + \frac{\Gamma'(1+\omega^{-1})}{\Gamma(\omega^{-1})} - Y_1(-1) - \frac{\bar{q}_*'(-1)}{P_0} - \eta(\alpha)\right\} - W(-1) \quad (4.7)$$

$$Y_1(p) = \frac{1}{2\omega i}\int_{-1-i\infty}^{-1+i\infty} [\ln F_2(t,0)]' \operatorname{ctg}\frac{\pi(t-p)}{\omega}\, dt$$

$$\eta(\alpha) = \alpha \operatorname{ctg} 2\alpha + {}^1/_3\alpha\cos 2\alpha + 2 + {}^1/_2 PP_0^{-1},\quad \alpha < {}^1/_2\pi$$

$$\eta(\alpha) = {}^3/_2 + PP_0^{-1},\quad \alpha = {}^1/_2\pi$$

In the case $\omega = -1$, we determine the constant $C_1$ from the equilibrium condition for a system of beams

$$\sum_{k=0}^{N-1}\int_0^\infty [N_k^1(r,0) - N_k^2(r,0) - q_k(r)]\, dr = P_0$$

which is, according to (2.1) and (2.7), equivalent to the condition $A(-1, 0) = P_0$. Hence, there also follows from (4.4)

$$C_1 = \pi^{1/2}(2\alpha)^{-1/2}(P_0 + {}^1/_2 P) - W(-1,0)$$

We determine the constant $C$ from the simple support condition for the ribbed plate, which is equivalent to the absence of a term proportional to $r^2$ in the asymptotic of the deflection $w(r, \gamma)$ for large $r$. By virtue of (2.6), we obtain for $N > 2$ $(\alpha < {}^1/_2\pi)$

$$C = -\frac{\sqrt{2}\,Y(-1)}{8DQ\sqrt{\pi\alpha}}\Big\{[\ln Q + \Gamma'(1) + Y_1(-1) + \alpha\operatorname{ctg}2\alpha +$$
$$ {}^1/_3\,\alpha\cos 2\alpha]\left[C_1 + W(-1) + {}^1/_2\, QY^{-1}(-1)\left(\frac{\pi}{2\alpha}\right)^{1/2} P\right] +$$
$$2C_1 + 2W(-1) + W_1(-1) +$$
$${}^1/_2\, QY^{-1}(-1)\left(\frac{\pi}{2\alpha}\right)^{1/2}[\bar{q}_*'(-1) + P]\Big\}$$

$$W_1(p) = \frac{1}{2i}\int_{-1-i\infty}^{-1+i\infty} f_1'(t)\sin^{-1}\pi(p-t)\, dt$$

The case $\omega = -1$, $N = 2$ is of only slight interest. It corresponds to the problem of bending of a plane reinforced by an infinite beam of constant stiffness. In Cartesian coordinates this problem is solved by elementary means.

Let $l = 1$. Then the highest term of the asymptotic (2.5) is generated by the residue of the integrand (2.6) in the second-order strip $p = 0$; the functions $\varphi_s(p, 1)$ are real on the imaginary axis and do not vanish for $p \neq 0$, the function $\varphi_1(p, 1)\varphi_2^{-1}(p, 1)$ has a simple pole at the point $p = 0$, therefore, $a = 0$ in (2.6), (2.8) and (4.1). By using (4.2) to confirm the conditions $\ln F_2(p, 1) \in H$ and $\operatorname{Ind} F_2(p, 1) = 0$, the solution of the problem (2.8) can be written for all $\omega \leqslant -1$

$$A(p, 1) = A_0(p, 1)\,[C_3 + \cos(\pi\omega^{-1}p)\, Z(p, 1)] \quad (4.8)$$
$$A_0(p, 1) = (p+1)(-\omega Q)^{-p/\omega}\Gamma(1 - \omega^{-1}p)\cos^{-1}({}^1/_2\pi\omega^{-1}p) \times X(p, 1)$$

The functions entering here are evaluated by means of (4.5) for $a = 0$, $l = 1$. Equating the factors for $r\ln r$ in (2.5) and in the residue (2.6) taken at the pole $p = 0$, we obtain

$$C_3 = M_0 A_0^{-1}(0, 1) - Z(0, 1) = \pi^{1/2}(-4\alpha\omega)^{-1/2}(M_0 + {}^1/_2 M) - W(0, 1) \quad (4.9)$$

Let us note that the transform found entirely determines the solution of the homogeneous problem of bending of a ribbed plate by moments $M_{01}$ and $M_{02}$; the quantities $M_{01}$ and $M_{02}$ doe not enter into the solution of the problem (2.2)−(2.5) for $l \neq 1$.

Let $l = 2, 3, \ldots, E[{}^1/_2(N+1)]$, then we have

$$A(p, l) = A_0(p, l)\cos(\pi\omega^{-1}p)\, Z(p, l) \quad (4.10)$$
$$A_0(p, l) = (-\omega Q)^{-p/\omega}(p+1)\,\Gamma\left(1 - \frac{p}{\omega}\right)\cos\frac{\pi p}{2\omega}\cos^{-1}\frac{\pi p}{\omega}\, X(p, l)$$
$$F_2(p, l) = -\varphi_1(p, l)\,\varphi_2^{-1}(p, l)\operatorname{ctg}(\pi\omega^{-1}p)$$

The remaining functions are taken from (4.5) for $a = 0$. As should have been expected, the solution (4.10) does not contain homogeneous solutions and new derivative constants.

From the definition of the transform (2.1) there results the identity $w_*(r, \gamma, N - l) = \overline{w_*(r, \gamma, l)}$. It hence follows that (2.6)−(4.10), (2.1) yield the solution of the boundary value problem (2.2)–(2.5) for all $l$. In the general case this solution is expressed by the sum $N$ of triple exponentially-convergent integrals; as is seen from the estimate (4.2), the rate of convergence diminishes as $|\omega|$ grows. To improve the convergence for large $|\omega|$ and to pass to the limit as $|\omega|\to\infty$, the solution of the homogeneous Barnes equation (2.8) is written in a different form /7/

$$\begin{aligned}
&A_0(p,0)=-\frac{\pi}{\omega}Q^{-p/\omega}(p+1)\Gamma\left(1-\frac{p}{\omega}\right)\Gamma^{-1}\left(\frac{p+1}{-\omega}\right)\times\\
&\frac{\cos\pi\omega^{-1}(p+1)}{\sin^2\pi\omega^{-1}(p+1)}T(p+1,\omega)X(p,0)\\
&A_0(p,1)=w^{-1}Q^{-p/\omega}p(p+1)\Gamma^2(-\omega^{-1}p)\sin^{-1}(\pi\omega^{-1}p)\times\\
&T(p,\omega)X(p,1)\\
&F_2(p,l)=\varphi_1(p,l)\varphi_2^{-1}(p,l)\operatorname{tg}\pi p,\ l=0,1\\
&A_0(p,l)={}^1/_2\omega^{-1}(\omega^2Q)^{-p/\omega}p(p+1)\Gamma^2(-\omega^{-1}p)\times\\
&\sin(2\pi\omega^{-1}p)T^{-1}(p,\omega)X(p,l)\\
&F_2(p,l)=\varphi_1(p,l)\varphi_2^{-1}(p,l)\operatorname{ctg}\pi p,\quad l=2,3,\ldots,E[{}^1/_2(N+1)]\\
&T(p,\omega)=\prod_{s=1}^{\infty}\Gamma\left(\frac{{}^1/_2-s-p}{\omega}\right)\Gamma\left(1-\frac{s-p}{\omega}\right)\Gamma^{-1}\left(\frac{s+p}{-\omega}\right)\times\\
&\Gamma^{-1}\left(1-\frac{s-{}^1/_2-p}{\omega}\right)\left(1-\frac{1}{2s}\right)^{1+2p\omega^{-1}}
\end{aligned}\tag{4.11}$$

If $N=2$, then instead of the first two formulas (4.11), simpler formulas can be used

$$A_0(p,0)=-\frac{\pi(p+1)}{\omega}(2Q)^{-p/\omega}\frac{\cos\pi\omega^{-1}(p+1)}{\sin^2\pi\omega^{-1}(p+1)}\times\Gamma\left(1-\frac{p}{\omega}\right)\Gamma^{-1}\left(\frac{p+1}{-\omega}\right)T\left(\frac{p+1}{2},\frac{\omega}{2}\right)\tag{4.12}$$

$$A_0(p,1)=\omega^{-1}p(p+1)(2Q)^{-p/\omega}\sin^{-1}(\pi\omega^{-1}p)T({}^1/_2p,{}^1/_2\omega)$$

**5.** As $\omega\to-\infty$, the beam stiffness $S(r)=Sr^{\omega+1}$ diminshes (increases) without limit for all $r>1$ $(r<1)$, and in the limit the system of infinite beams degenerates into a finite linear star stamp $0\leqslant r\leqslant 1$, $\theta=2\alpha k$ (Fig.2). The fundamental conditions (1.3) go over into mixed conditions here

$$\partial^2w_k^1(r,0)/\partial r^2=0,\qquad 0\leqslant r\leqslant 1;\ N_k^1(r,0)-N_k^2(r,0)=q_k(r),\ 1<r<\infty\tag{5.1}$$

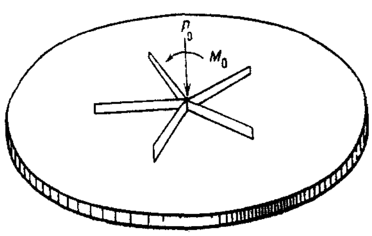


Fig.2

The remaining conditions do not change.

The Wiener–Hopf method could be applied to this problem after it had been reduced to a boundary value problem for the transform $w_*(r,\gamma,l)$ of the type (2.2)–(2.5). Following /7/ here, we again write the solution down for the case $q_k(r)\equiv 0$, $k=0.1,\ldots,N-1$, by passing to the limit as $\omega\to-\infty$ in the results obtained. The form of the solution of the problem (5.1) in the transforms $w_*^j(r,\gamma,l)$ here conserves its previous form (2.6), (2.7). According to (4.11) $T(p,-\omega)=\pi\Gamma(p+{}^1/_2)\Gamma^{-1}(p+1)$. Hence, we also obtain from the solution (4.3)–(4.5), (4.8), (4.11):

$$\begin{aligned}
&A(p,0)=\pi^{-3/2}\Gamma(p+{}^3/_2)\Gamma^{-1}(p+2)X(p,0)[C_1+C_2(p+1)]\\
&A(p,1)=\pi^{-3/2}C_3(p+1)\Gamma(p+{}^1/_2)\Gamma^{-1}(p+1)X(p,1)\\
&A(p,l)\equiv 0,\ l=2,3,\ldots,E[{}^1/_2(N+1)]\\
&X(p,l)=Y(p,l)F_2^{-1}(p,l),\ \operatorname{Re}p<a\\
&X(p,l)=Y(p,l),\ \operatorname{Re}p\geqslant a,\ a=-\delta_{0l}\\
&Y(p,l)=\exp\left[\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\ln F_2(t,l)\frac{dt}{t-p}\right]
\end{aligned}\tag{5.2}$$

where the functions $F_2(p, l)$ are determined by (4.11).

The arbitrary constants are found from condition (2.5), as in Sect.4, by using an expansion in residues

$$C_1 = \pi^{3/2}\alpha^{-1/2}P_0 [Y(-1, 0)]^{-1}$$
$$C_2 = C_1 \{\Gamma'(1) - \pi^{-1/2}\Gamma'(1/2) - Y_1(-1) - \eta(\alpha)\}$$
$$C_3 = \pi^{3/2}(2\alpha)^{-1/2}M_0 \quad (5.3)$$

Passing to the limit in $\omega$ in (4.12) for $N=2$, we obtain the simpler formulas

$$A(p, 0) = \pi^{-1/2}\Gamma(1 + 1/2p)\Gamma^{-1}(3/2 + 1/2p)[C_1 + C_2(p+1)]$$
$$A(p, 1) = C_3\pi^{-1/2}(p+1)\Gamma(1/2 + 1/2p)\Gamma^{-1}(1 + 1/2p)$$
$$C_1 = \pi P_0,\ C_2 = 1/2C_1[\Gamma'(1) - \pi^{-1/2}\Gamma'(1/2) - 3],\ C_3 = \pi M_{01} \quad (5.4)$$

The solution (2.6), (2.7), (5.4) corresponds to impression in a plate, simply supported on a circumference $r = R, R \gg 1$ of a linear stamp $0 \leqslant r \leqslant 1,\ \theta = 0,\ \theta = \pi$.

The passage from the transform (2.6), (2.7), (5.4) to the solution of the problem (1.1), (1.2), (5.1), (1.4), (1.5) is accomplished in the $k$-th sector by the inversion formula (2.1).

Let us clarify the behavior of the contact pressure $\Delta N_k(r) = N_k^1(r, 0) - N_k^2(r, 0)$ at the edge of the $k$-th rib of a star stamp. According to (2.1), (2.7), the double transform (discrete and Mellin transformations) of the function $\Delta N_k(r)$ equals $A(p, l)$. By virtue of (5.2), for large $p, \operatorname{Re} p > a$, we have

$$A(p, 0) \sim \pi^{-3/2}(C_1p^{-1/2} + C_2p^{1/2})$$
$$A(p, 1) \sim C_3\pi^{-3/2}p^{1/2},\ A(p, l) \equiv 0 \quad (l = 2, \ldots, E[1/2(N+1)]) \quad (5.5)$$

There is an asymptotic connnection between the function $f(r)$ and its transform $F^+(p)$. If

$$F^+(p) = \int_0^1 f(r) r^p\, dr \sim Ap^{-\eta-1}, \quad \operatorname{Re} p > a, \quad -1 < \eta < 0$$

then $f(r) \sim A\Gamma^{-1}(\eta+1)(1-r)^\eta$ as $r \to 1-0$. Using this connection for the original function $\Delta N_*(r, l)$ and differentiating the result according to (2.1), (5.5) with the identity $\Delta N_*(r, N-l) = \overline{\Delta N_*(r, l)}$ taken into account, we obtain $(k = 0, 1, \ldots, N-1)$

$$\Delta N_k(r) \sim \frac{1}{N\pi^2}\left[\frac{C_1}{(1-r)^{1/2}} - \frac{C_2 + 2\operatorname{Re}(C_3e^{-2i\alpha k})}{2(1-r)^{3/2}}\right], \quad r \to 1-0$$

It can be shown that the elastic strain energy of the plate is here bounded in the neighborhood of the rib ends.

If $N=2$, then the function $\Delta N_k(r)$ is expressed in radicals in the whole band $(0, 1)$. Indeed, for $r < 1$ according to the Cauchy theorem the contour integrals in the formula

$$\Delta N_k(r) = \frac{1}{4\pi i}\int_L [A(p, 0) + (-1)^k A(p, 1)]\, r^{-p-2}dp; \quad k = 0, 1$$

can, according to (5.4), be replaced by series of residues at the poles of the Gamma functions $\Gamma(1 + 1/2p)$ and $\Gamma(1/2 + 1/2p)$ starting with the point $p = -2$. Summing these series by means of formulas 9.03 and 9.05 from /13/, and taking account of the equality $\Gamma'(1) - \pi^{1/2}\Gamma'(1/2) = 2\ln 2$, we obtain

$$\Delta N_k(r) = \frac{P_0}{\pi}\left[\frac{1}{(1-r^2)^{1/2}} - \frac{2\ln 2 - 3}{2(1-r^2)^{3/2}}\right] - \frac{(-1)^k M_{01} r}{(1-r^2)^{3/2}}$$

For $M_{01} = 0$ this agrees with the result in /14/ found by different means.

**6.** The constraint $N \geqslant 2$ was imposed in Sect.4. If $N = 1$ then the solution can be obtained by following /7/ and not applying the discrete transformation. The results of Sects. 1–5 can also be used by changing the function $F_2(p, l)$ somewhat since the real zeroes of the function $\varphi_s(p, l)$ are distributed differently for $N = 1$.

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